EXTENDED ELECTRODYNAMICS:

I. Basic Notions, Principles and Equations

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Abstract

This paper aims to present an elaborate view on the motivation and realization of the idea to extend Maxwell's electrodynamics to *Extended Electrodynamics* in a reasonable and appropriate way in order to make it possible to describe electromagnetic (3+1)-soliton-like objects in vacuum and in the presence of continuous media (external fields) [15]-[20], exchanging energy-momentum with the electromagnetic field.

1. Preliminary notes

Classical Electrodynamics (CED) [1] is probably the most fascinating and complete part of the Classical Field Theory. Intuition, free thought, perspicuity and research skill of many years finally brought about the synthesis of experiment and theory, of physics and mathematics, which we have been calling for short the *Maxwell equations* for the duration of a century and a half. From the beginning of the second half of the 19th century till its end these equations turned from abstract theory into daily practice, as they are today. Their profound study during the first half of the 20th century brought forward a new theoretical concept in physics known as *relativism*. Brave and unprejudiced scientists enriched and widened the synthesis achieved through the Maxwell equations, and created a new synthesis known briefly as *quantum electrodynamics*. Every significant scientific breakthrough is based on two things: *respect for the workers and their work and respect for the truth.* "May everyone be respected as a personality, and nobody as an idol" one of the old workers used to say. We may paraphrase that saying: "may every scientific truth be respected, but no one be turned into dogma".

In our attempt to extend CED we tried to follow the values this creed teaches, as far as our humble abilities allow us to. Together with the analysis of the classical electrodynamics and the discrete conception for the structure of the electromagnetic field, the path followed brought us to the conclusion that a *soliton-like solution of appropriate non-linear equations characterized by an intrinsic periodical process is the most adequate mathematical model of the basic structural unit of the field - the photon.* The fact that neither the Maxwell equations nor the quantum electrodynamics offer the appropriate tools to find such solutions, unmistakably emphasizes the necessity to look for new equations. We decided not to follow the usual way for nonlinearization of CED [2], because, from our point of view, it does not comprise sufficiently fruitful and new ideas. Other attempts in this direction one may find in [3]-[12].

The leading physical ideas in our approach were the dual ("electromagnetic") nature of the field on the one hand and the local Energy-Momentum Conservation Laws on the other. The realization that every such soliton-like solution determines in an invariant way its own *scale factor*, as well as the suitable interpretation of the famous formula of Planck for the relation between the full energy E of the photon and the frequency $\nu = 1/T$ of the beforementioned periodical process, which we prefer to write

down as $h = E.T$, helped us to formulate the rules for separating the realistic soliton-like models of the photon from among the rest. The resulting soliton-like solutions [19] possess all integral qualities of the photon, as described by quantum electrodynamics, but also a structure, organically tied to an intrinsic periodical process, which in its turn generates an intrinsic mechanical angular momentum - spin (helicity). We consider this soliton- like oscillating non-linear wave much more clear and understandable than the *ambiguous "particle-wave" duality*.

The dual 2-component nature of the field predetermined to a great extent the generalization of the equations in the case of an interaction with another continuous physical object, briefly called medium [20]. The proposed physical interpretation of the classical Frobenius equations for complete integrability [13],[14] of a system of non-linear partial differential equations as a *criterion for the absence of dissipation*, turned out to be relevant and was effectively used. The fruitfulness of the new non-linear equations is clearly shown by the family of solutions, giving (3+1)-dimensional interpretation of all (1+1)-dimensional 1-soliton and multisoliton solutions [20]. We note that making use of differential geometry proved extremely useful.

This paper follows the main track of our efforts to build a clear and consecutive picture of motivations and theoretical results. While most of the solutions to the new nonlinear equations we have found have been already published [15]-[20], a consecutive and well motivated reasoning, leading to the new equations is still missing. So, the stress in this paper is laid on the conceptual and generic framework. The purpose is to bring the reader to the conviction that the extension of CED, developed by us during the last 10 years, is necessary and is physically and mathematically quite natural. We do not present solutions to equations. We set the problem to build a description of electromagnetic soliton-like objects, then we consider step by step physical arguments and try to find the corresponding step by step mathematical adequacies. Our belief is that if the steps are in the right direction the positive results come inevitably. In its turn every new positive result (solution or relation), suggests new insights and invites to search for new more definite quantities and relations. This, step by step creative process, brought us to the today's motivational look on what we call *Extended Electrodynamics* (EED). It worths from time to time to reconsider and re-estimate the importance and significance of any reason and idea been used, because this gives birth to new reasons and ideas and helps to sift out the basic and meaningful from the occasional and nonsignificant.

2. Physical conception for the electromagnetic field in Extended Electrodynamics

As it is well known, the mathematical models of the real vacuum electromagnetic fields in Classical Electrodynamics are "infinite", or if they are finite, they are strongly time-unstable. These models are not consistent with a number of experimental facts. A deeper analysis of CED, carried out in the first third of this century, resulted in the new conception for a discrete character of the field. This physical understanding of the field is the true foundation of EED and it shows clearly the principal differences between CED and EED. For the sake of clarity we shall formulate our point of view more explicitly.

The electromagnetic field in vacuum is of discrete character and consists of single, not-interacting (or very weakly interacting) finite objects, called photons. All photons move uniformly as a whole by the same velocity 'c', carry finite energy 'E', momentum '$\mathbf{p}$' and intrinsic angular momentum. These features imply structure and internal periodic process of period 'T', which may be different for the various photons. The quantity 'E.T', called "elementary action", has the same value for all photons and is numerically equal to the Planck constant 'h'. The invariance of 'c' and 'h' means nondistinguishability of the photons, considered as invariant objects. The integral value of the intrinsic angular momentum is equal to '$\pm h$'. For the topology of the 3-dimensional region, occupied by the photon at any moment, there are no experimental data, so it is desirable that the model-solutions to allow arbitrary initial data.

We'd like to stress the following: EED considers photons as *real finite objects*, and *not as convenient theoretical concepts*, and it aims to build adequate mathematical models of these entities. So, the first important problem is to point out the algebraic character of the mathematical object describing a *single* photon. The corresponding generalization for a number of photons is then easily done.

3. Choice of the modelling mathematical object

According to the non-relativistic formulation of CED the electromagnetic field has two aspects: "electric" and "magnetic". These two aspects of the field are described by two 1-forms on $\mathcal{R}^3$ and a parametric dependence on

time is possible: the electric field E and the magnetic field B. The following considerations will bring us to the conclusion that these two fields can be considered as two *vector components* of a new object, 1-form Ω, *taking values in a real 2-dimensional vector space*, naturally identified with $\mathcal{R}^2$. In fact, let's consider the question: do there exist constants (a, b, m, n), such that the linear combinations

$$E' = aE + bB, \ B' = mE + nB$$

give again a solution? The answer to this question is positive iff $m = -b$, $n = a$. The new solution will have energy density w' and momentum $\mathbf{S}'$ as follows:

$$w' = \frac{1}{8\pi}\Big((E')^2 + (B')^2\Big) = \frac{1}{8\pi}(a^2 + b^2)\Big(E^2 + B^2\Big),$$

$$\mathbf{S}' = (a^2 + b^2)\frac{c}{4\pi}E \times B.$$

Obviously, the new and the old solutions will have the same energy and momentum if $a^2 + b^2 = 1$.

This simple but important observation shows that besides the usual linearity, Maxwell's equations admit also "cross"-linearity, i.e. linear combinations of E and B of a definite kind define new solutions. Therefore, the difference between the electric and magnetic fields becomes non-essential. The important point is that with every solution (E, B) of Maxwell's equations a 2-dimensional real vector space, spanned by the couple (E, B), is associated, and the linear transformations, which transform solutions into solutions, are given by matrices of the kind

$$\begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}.$$

If these matrices are unimodular, i.e. if $a^2+b^2 = 1$, the energy and momentum do not change. It is well known that matrices of this kind do not change the canonical complex structure J in $\mathcal{R}^2$.(Recall that if the canonical basis of $\mathcal{R}^2$ is denoted by (e_1, e_2) then J is defined by $J(e_1) = e_2$, $J(e_2) = -e_1$.)

The above remarks suggest to consider E and B as two vector-components of an $\mathcal{R}^2$-valued 1-form Ω:

$$\Omega = E \otimes e_1 + B \otimes e_2.$$

So, the current $\mathbf{j}$ becomes 1-form $\mathcal{J} = \mathbf{j} \otimes e_1$ with values in $\mathcal{R}^2$, and the charge density becomes a function $\mathcal{Q} = \rho \otimes e_1$ with values in $\mathcal{R}^2$. Maxwell's equations

$$\frac{1}{c}\frac{\partial E}{\partial t} = rotB - \frac{4\pi}{c}\mathbf{j}, \quad divB = 0, \tag{1}$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = -rotE, \qquad divE = 4\pi\rho. \tag{2}$$

take the form

$$\frac{1}{c}\frac{\partial \Omega}{\partial t} = -\frac{4\pi}{c}\mathcal{J} - *\mathbf{d}J(\Omega), \quad \delta\Omega = 4\pi\mathcal{Q}, \tag{3}$$

where $J(\Omega) = E \otimes J(e_1) + B \otimes J(e_2) = E \otimes e_2 - B \otimes e_1$ and $\mathbf{d}$ is the exteriour derivative. Note that according to the sense of the concept of current in CED and because of the lack of magnetic charges, it is necessary to exist a basis in $\mathcal{R}^2$, in which $\mathcal{J}$ and $\mathcal{Q}$ to have components only along e_1. Nevertheless, this point of view shows that even at this non-relativistic level the separation of the EM-field to "electric" and "magnetic" is not adequate to the real situation. The mathematical object Ω unifies and, at the same time, distinguishes the two sides of the field: there is a basis in $\mathcal{R}^2$, in which the electric and magnetic components are delimited, but in an arbitrary basis the two components mix (superimpose), so the difference between them is deleted.

In the relativistic formulation of CED the difference between the electric and magnetic components of the field is already quite conditional, and from invariant-theoretical point of view there is no difference. However, the 2-component character of the field acquires a new meaning and manifests itself differently. In order to come to this we make the following considerations.

As we mentioned above, some linear combinations of the electric and magnetic fields generate a new solution to Maxwell's equations. In particular, the transformation, defined by the matrix

$$\begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix},$$

defining a complex structure in $\mathcal{R}^2$, transforms a solution of the kind $(E, 0)$ into a new solution of the kind $(0, E)$ and a solution of the kind $(0, B)$ into a solution of the kind $(-B, 0)$, i.e. the electric component into magnetic and vice versa. This observation draws our attention to looking for a complex structure $J, J^2 = -id$ in the bundle of 2-forms on the Minkowski space M,

such that if F presents the first component of the field, then $J(F)$ to present the second component of the same field. Such complex structure really exists, in fact, it coincides with the restriction of the Hodge $*$-operator, defined by the pseudometric η through the equation

$$\alpha \wedge *\beta = -\eta(\alpha,\beta)\sqrt{|\eta|}dx \wedge dy \wedge dz \wedge d\xi = -\eta(\alpha,\beta)\omega_o, \ \xi = ct,$$

to the space of 2-forms: $**_2 = -id_{\Lambda^2(M)}$. So, the non-relativistic vector components (E, B) are replaced by the relativistic vector components $(F, *F)$. The following considerations support also such a choice.

The relativistic Maxwell's equations in vacuum $\mathbf{d}F = 0$, $\mathbf{d} * F = 0$ are, obviously, invariant with respect to the interchange $F \to *F$. Moreover, if F is a solution, then an arbitrary linear combination $aF + b * F$ is again a solution. More generally, if $(F, *F)$ defines a solution, then the transformation $(F, *F) \to (aF + b * F, mF + n * F)$ defines a new solution for an arbitrary matrix

$$\left\| \begin{matrix} a & m \\ b & n \end{matrix} \right\|.$$

Now, using the old notation Ω for the new object $\Omega = F \otimes e_1 + *F \otimes e_2$, Maxwell's equations are written down as $\mathbf{d}\Omega = 0$, or equivalently $\delta\Omega = 0$, where the coderivative operator $\delta = *\mathbf{d}*$ is just the (minus) *divergence*: $(\delta F)_\mu = -\nabla_\nu F^\nu_\mu$. Clearly, an arbitrary linear transformation of the basis (e_1, e_2) keeps Ω as a solution.

Recall now the energy-momentum tensor in CED, defined by

$$Q^\nu_\mu = \frac{1}{4\pi}\left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^\nu_\mu - F_{\mu\sigma}F^{\nu\sigma}\right] = \frac{1}{8\pi}\left[-F_{\mu\sigma}F^{\nu\sigma} - (*F)_{\mu\sigma}(*F)^{\nu\sigma}\right]. \quad (4)$$

It is quite clearly seen, that F and $*F$ participate in the same way in Q^ν_μ, and the full energy-momentum densities of the field are obtained through summing up the energy-momentum densities, carried by F and $*F$. Since the two expressions $F_{\mu\sigma}F^{\nu\sigma}$ and $(*F)_{\mu\sigma}(*F)^{\nu\sigma}$ are not always equal, the distribution of energy-momentum between F and $*F$ may change in time, i.e. energy-momentum may be transferred from F to $*F$, and vice versa. So we may interpret this phenomenon as a special kind of interaction between F and $*F$, responsible for some internal redistribution of the field energy. Now, in vacuum it seems naturally to expect, that the energy-momentum, carried from F to $*F$ in a given 4-volume, is the same as that, carried from $*F$ to F in the same volume. However, in presence of an active external

field, exchanging energy-momentum with Ω, it is hardly reasonable to trust the same expectation just because of the specific structure the external field (medium) may have. So, the external field (further any such external field will be called also *medium* for short) may exchange energy-momentum preferably by F or $*F$, as well as it may support the internal redistribution of the field energy-momentum between F and $*F$, favouring F or $*F$. From the explicit form of the energy-momentum tensor it is seen that the field may participate in this exchange by means of any of the two terms $F_{\mu\sigma}F^{\nu\sigma}$ and $(*F)_{\mu\sigma}(*F)^{\nu\sigma}$. Moreover, for the divergence of the energy-momentum tensor we easily obtain

$$\nabla_\nu Q_\mu^\nu = \frac{1}{4\pi}\Big[F_{\mu\nu}(\delta F)^\nu + (*F)_{\mu\nu}(\delta * F)^\nu\Big]. \tag{5}$$

It is clearly seen that the quantities of energy-momentum, which any of the two components F and $*F$ may exchange in a unit 4-volume are *invariantly* separated and given respectively by

$$F_{\mu\nu}(\delta F)^\nu, \quad (*F)_{\mu\nu}(\delta * F)^\nu.$$

But, in CED the exchange through $*F$ *is forbidden*, the expression $(*F)_{\mu\nu}(\delta * F)^\nu$ is always, in *all media*, equal to zero. This comes from the unconditional assumption, that the Faraday's induction law is universally true. Of course, we do not reject the existence of media, not allowing energy-momentum exchange through $*F$, but we do not share the opinion that all media behave in this same way. On the other hand, in case of vacuum, we can not delimit F from $*F$, these are two solutions of the same equation and it is all the same which one will be denoted by F (or $*F$), i.e. CED does not give an intrinsic criterion for a respective choice. Only in regions with non-zero free charges and currents, when $\mathbf{d}F = 0$ and $\delta F = 4\pi j \neq 0$, the choice can be made, but this presupposes (postulates) that the field is able to interact, i.e. to exchange energy-momentum, *only* with charged particles, i.e. through F. This postulate we can not assume ad hoc.

Having in view these considerations we assume the following postulate in EED in order to specify the algebraic character of the modelling mathematical object:

In EED the electromagnetic field is described by a 2-form Ω, defined on the Minkowski space-time and valued in a real 2-dimensional vector space $\mathcal{W}$

and such, that there is a basis (e_1, e_2) *of* $\mathcal{W}$ *in which* Ω *takes the form*

$$\Omega = F \otimes e_1 + *F \otimes e_2. \tag{6}$$

Since $\mathcal{W}$ is isomorphic to $\mathcal{R}^2$, further we shall write only $\mathcal{R}^2$ and all relations obtained can be carried over to $\mathcal{W}$ by means of the corresponding isomorphism. In particular, every $\mathcal{W}$ will be considered as being provided with a complex structure J, so, the group of automorphisms of J is defined. Our purpose now is to prove that the set of 2-forms of the kind (6) is stable under the invariance group of J.

First we note, that the equation $aF + b * F = 0$ requires $a = b = 0$. In fact, if $a \neq 0$ then $F = -\frac{b}{a} * F$. From $aF + b * F = 0$ we get $a * F - bF = 0$ and substituting F, we obtain $(a^2 + b^2) * F = 0$, which is possible only if $a = b = 0$ since $*F \neq 0$. In other words, F and $*F$ are linearly independent. Let now (k_1, k_2) be another basis of $\mathcal{R}^2$ and consider the 2-form $\Psi = G \otimes k_1 + *G \otimes k_2$. We express (k_1, k_2) through (e_1, e_2) and take in view what we want:

$$G \otimes k_1 + *G \otimes k_2 = G \otimes (ae_1 + be_2) + *G \otimes (me_1 + ne_2) =$$

$$= (aG + m*G) \otimes e_1 + (bG + n*G) \otimes e_2 = (aG + m*G) \otimes e_1 + *(aG + m*G) \otimes e_2.$$

Consequently, $bG + n * G = a * G - mG$, i.e. $(b + m)G + (n - a) * G = 0$, which requires $m = -b,\ n = a$, i.e. the transformation matrix is

$$\begin{Vmatrix} a & -b \\ b & a \end{Vmatrix}.$$

Besides, if Ω_1 and Ω_2 are of the kind (6), it is easily shown that the linear combination $\lambda\Omega_1 + \mu\Omega_2$ is of the same kind (6). These results show that the 2-forms of the kind (6) form a stable with respect to the automorphisms of $(\mathcal{R}^2, J)$ subspace of the space $\Lambda^2(M, \mathcal{R}^2)$. Moreover, if the component F is chosen, then the basis (e_1, e_2) is uniquely determined. In other words, every Ω of the kind (6) allows various representations of this kind, but every component F determines unique basis $(e_1(F), e_2(F))$, and so 2 subspaces $\{e_1\}$ and $\{e_2\}$ of $\mathcal{R}^2$. (Further the argument (F) of the corresponding basis is omitted).

In order to separate the class of bases we are going to use, first we recall the product of 2 vector valued differential forms. If Φ and Ψ are respectively p and q forms on the same manifold N, taking values in the vector spaces

W_1 and W_2 with corresponding bases $(e_1, ..., e_m)$ and $(k_1, ..., k_n)$, and $\varphi : W_1 \times W_2 \to W_3$ is a bilinear map into the vector space W_3, then a $(p+q)$-form $\varphi(\Phi, \Psi)$ on N with values in W_3 is defined by

$$\varphi(\Phi, \Psi) = \sum_{i,j} \Phi^i \wedge \Psi^j \otimes \varphi(e_i, k_j).$$

In particular, if $W_1 = W_2$ and $W_3 = \mathcal{R}$, and the bilinear map is scalar (inner) product g, we get

$$\varphi(\Phi, \Psi) = \sum_{i,j} \Phi^i \wedge \Psi^j g_{ij}.$$

Let now X and Y be 2 arbitrary vector fields on the Minkowski space M, Ω be of the kind (6), $Q_{\mu\nu}$ be the energy tensor in CED and g be the canonical inner product in $\mathcal{R}^2$. Then the class of bases we shall use will be separated by the following equation

$$Q_{\mu\nu} X^\mu Y^\nu = \frac{1}{2} * g\Big(i(X)\Omega, *i(Y)\Omega\Big). \tag{7}$$

We develop the right hand side of this equation and obtain

$$\frac{1}{2} * g\Big(i(X)\Omega, *i(Y)\Omega\Big) =$$

$$\frac{1}{2} * g\Big(i(X)F \otimes e_1 + i(X) * F \otimes e_2, *i(Y)F \otimes e_1 + *i(Y) * F \otimes e_2\Big) =$$

$$= \frac{1}{2} * \Big[\Big(i(X)F \wedge *i(Y)F\Big)g(e_1, e_1) + \Big(i(X)F \wedge *i(Y) * F\Big)g(e_1, e_2) +$$

$$+ \Big(i(X) * F \wedge *i(Y)F\Big)g(e_2, e_1) + \Big(i(X) * F \wedge *i(Y) * F\Big)g(e_2, e_2)\Big] =$$

$$= -\frac{1}{2} X^\mu Y^\nu \Big[F_{\mu\sigma} F_\nu^\sigma g(e_1, e_1) + (*F)_{\mu\sigma}(*F)_\nu^\sigma g(e_2, e_2) +$$

$$+ \Big(F_{\mu\sigma}(*F)_\nu^\sigma + (*F)_{\mu\sigma} F_\nu^\sigma\Big)g(e_1, e_2)\Big] = -\frac{1}{2} X^\mu Y^\nu \Big[F_{\mu\sigma} F_\nu^\sigma + (*F)_{\mu\sigma}(*F)_\nu^\sigma\Big].$$

In order this relation to hold it is necessary to have

$$g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_1, e_2) = 0,$$

i.e., we are going to use *orthonormal* bases. So, the stability group of the subspace of forms of the kind (6) is reduced to $SO(2)$ or $U(1)$. So, in

this approach, the group $SO(2)$ appears in a pure algebraic way, while in the gauge interpretation of CED this group is associated with the equation $\mathbf{d}F = 0$, i.e. with the traditional and not shared by us understanding, that the EM-field can not exchange energy-momentum with any medium through $*F$.

4. Differential equations for the field

We proceed to the main purpose, namely, to write down differential equations for our object Ω, which was chosen to model the EM-field. We shall work in the orthonormal basis (e_1, e_2), where the field has the form (6). The two vectors of this basis define two mutually orthogonal subspaces $\{e_1\}$ and $\{e_2\}$, such that the space $\mathcal{R}^2$ is a direct sum of these two subspaces: $\mathcal{R}^2 = \{e_1\} \oplus \{e_2\}$. So, we have the two projection operators $\pi_1 : \mathcal{R}^2 \rightarrow \{e_1\}$, $\pi_2 : \mathcal{R}^2 \rightarrow \{e_2\}$. These two projection operators extend to projections in the $\mathcal{R}^2$-valued differential forms on M:

$$\pi_1\Omega = \pi_1(\Omega^1 \otimes k_1 + \Omega^2 \otimes k_2) = \Omega^1 \otimes \pi_1 k_1 + \Omega^2 \otimes \pi_1 k_2 =$$

$$= \Omega^1 \otimes \pi_1(ae_1 + be_2) + \Omega^2 \otimes \pi_1(me_1 + ne_2) = (a\Omega^1 + m\Omega^2) \otimes e_1.$$

Similarly,

$$\pi_2\Omega = (b\Omega_1 + n\Omega_2) \otimes e_2.$$

In particular, if Ω is of the form (6), then

$$\pi_1(F \otimes e_1 + *F \otimes e_2) = F \otimes e_1, \ \ \pi_2(F \otimes e_1 + *F \otimes e_2) = *F \otimes e_2.$$

Let now our EM-field Ω propagates in a region, where some other continuous physical object (external field, medium) also propagates and exchanges energy-momentum with Ω. We are going to define explicitly the local law this exchange obeys.

First we note, that the external field is described by some mathematical object(s). From this mathematical object, following definite rules, reflecting the specific situation under consideration, a new mathematical object $\mathcal{A}_i$ is constructed and this new mathematical object participates directly in the exchange defining expression. The EM-field participates in this exchange defining expression directly through Ω, and since Ω takes values in $\mathcal{R}^2$, then $\mathcal{A}_i$ must also take values in $\mathcal{R}^2$.

We make now two preliminary remarks. First, all operators, acting on the usual differential forms, are naturally extended to act on vector valued differential forms according to the rule $D \rightarrow D \times id$. In particular,

$$*\Omega = *(\sum_i \Omega^i \otimes e_i) = \sum_i (*\Omega^i) \otimes e_i,\ \mathbf{d}\Omega = \mathbf{d}(\sum_i \Omega^i \otimes e_i) = \sum_i (\mathbf{d}\Omega^i) \otimes e_i,$$

$$\delta\Omega = \delta(\sum_i \Omega^i \otimes e_i) = \sum_i (\delta\Omega^i) \otimes e_i.$$

Second, in view of the importance of the expression (5) for the divergence of the CED energy-momentum tensor, we give its explicit deduction. Recall the following algebraic relations on the Minkowski space:

$$\delta_p = (-1)^p *^{-1} \mathbf{d}* = *\mathbf{d}*,\ \delta*_p = *\mathbf{d}_p\ for\ p = 2k+1,\ \delta*_p = -*\mathbf{d}_p\ for\ p = 2k. \tag{8}$$

If α is a 1-form and F is a 2-form, the following relations hold:

$$*\,(\alpha \wedge *F) = -\alpha^\mu F_{\mu\nu} dx^\nu = *\,[(*F) \wedge *(*\alpha)] = \frac{1}{2}(*F)^{\mu\nu}(*\alpha)_{\mu\nu\sigma} dx^\sigma. \tag{9}$$

In particular,

$$*(F \wedge *\mathbf{d}F) = \frac{1}{2} F^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma} dx^\sigma = *[\delta * F \wedge *(*F)] = (*F)_{\mu\nu}(\delta * F)^\nu dx^\mu.$$

Having in view these relations, we obtain consecutively:

$$\nabla_\nu Q^\nu_\mu = \nabla_\nu \left[\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta^\nu_\mu - F_{\mu\sigma} F^{\nu\sigma}\right] =$$

$$= \frac{1}{2} F^{\alpha\beta} \nabla_\nu F_{\alpha\beta} \delta^\nu_\mu - (\nabla_\nu F_{\mu\sigma}) F^{\nu\sigma} - F_{\mu\sigma} \nabla_\nu F^{\nu\sigma} =$$

$$= \frac{1}{2} F^{\alpha\beta}[(\mathbf{d}F)_{\alpha\beta\mu} - \nabla_\alpha F_{\beta\mu} - \nabla_\beta F_{\mu\alpha}] - (\nabla_\nu F_{\mu\sigma}) F^{\nu\sigma} - F_{\mu\sigma} \nabla_\nu F^{\nu\sigma} =$$

$$= \frac{1}{2} F^{\alpha\beta}(\mathbf{d}F)_{\alpha\beta\mu} - F_{\mu\sigma} \nabla_\nu F^{\nu\sigma} = -(*F)_{\mu\nu} \nabla_\sigma (*F)^{\sigma\nu} - F_{\mu\nu} \nabla_\sigma F^{\sigma\nu} =$$

$$= (*F)_{\mu\nu}(\delta * F)^\nu + F_{\mu\nu}(\delta F)^\nu.$$

Let now our field Ω interact with some other field. This interaction, i.e. energy-momentum exchange, is performed along 3 "channels". The first 2 channels are defined by the 2 (equal in rights) components F and $*F$ of

Ω. This exchange is *real* in the sense, that some part of the EM-energy-momentum may be transformed into some other kind of energy-momentum and absorbed by the external field or dissipated. Since the two components F and $*F$ are equal in rights it is naturally to expect that the corresponding 2 terms, defining the exchange in a unit 4-volume, will depend on F and $*F$ similarly. The above expression for $\nabla_\nu Q^\nu_\mu$ gives the two 1-forms

$$F_{\mu\nu}(\delta F)^\nu dx^\mu, \quad (*F)_{\mu\nu}(\delta * F)^\nu dx^\mu$$

as natural candidates for this purpose. As for the third channel, it takes into account a possible influence of the external field on the intra-field exchange between F and $*F$, which occurs without absorbing of energy-momentum by the external field. The natural candidate, describing this exchange is, obviously, the expression

$$F_{\mu\nu}(\delta * F)^\nu dx^\mu + (*F)_{\mu\nu}(\delta F)^\nu dx^\mu.$$

It is important to note, that these three channels are independent in the sense, that any of them may occur without taking care if the other two work or don't work. A natural model for such a situation is a 3-dimensional vector space K, where the three dimensions correspond to the three exchange channels. The non-linear exchange law requires some K-valued non-linear map. Since our fields take values in $\mathcal{R}^2$ this 3-dimensional space must be constructed from $\mathcal{R}^2$ in a natural way. Having in view the bilinear character of $\nabla_\nu Q^\nu_\mu$ it seems naturally to look for some bilinear construction with the properties desired. These remarks suggest to choose for K the *symmetrized tensor product* $Sym(\mathcal{R}^2 \otimes \mathcal{R}^2) \equiv \mathcal{R}^2 \vee \mathcal{R}^2$. So, from the point of view of the EM-field, the energy-momentum exchange term should be written in the following way:

$$* \vee (\delta\Omega, *\Omega). \tag{10}$$

In fact, in the corresponding basis (e_1, e_2) we obtain

$$* \vee (\delta\Omega, *\Omega) = * \vee (\delta F \otimes e_1 + \delta * F \otimes e_2, *F \otimes e_1 + ** F \otimes e_2) =$$

$$= *(\delta F \wedge *F) \otimes e_1 \vee e_1 + *(\delta * F \wedge ** F) \otimes e_2 \vee e_2 + *(-\delta F \wedge F + \delta * F \wedge *F) \otimes e_1 \vee e_2$$

$$= F_{\mu\nu}(\delta F)^\nu dx^\mu \otimes e_1 \vee e_1 + (*F_{\mu\nu})(\delta * F)^\nu dx^\mu \otimes e_2 \vee e_2 +$$

$$+ [F_{\mu\nu}(\delta * F)^\nu dx^\mu + (*F_{\mu\nu})\delta F^\nu dx^\mu] \otimes e_1 \vee e_2.$$

This expression determines how much of the EM-field energy-momentum may be carried irreversibly over to the external field (the first and the second terms) and how much may be redistributed between F and $*F$ by virtue of the external field influence in a unit 4-volume.

Now, this same quantity of energy-momentum has to be expressed by new terms, in which the external field "agents" should participate. Let's denote by Φ the first agent, interacting with $\pi_1\Omega$, and by Ψ the second agent, interacting with $\pi_2\Omega$. Since the corresponding two exchanges are independent, we may write the exchange term in the following way:

$$\vee(\Phi, *\pi_1\Omega) \ + \ \vee(\Psi, *\pi_2\Omega). \tag{11}$$

According to the local energy-momentum conservation law these two quantities have to be equal, so we obtain

$$\vee(\delta\Omega, *\Omega) = \vee(\Phi, *\pi_1\Omega) \ + \ \vee(\Psi, *\pi_2\Omega). \tag{12}$$

This is the basic relation in EED. It contains the basic differential equations for the EM-field components and requires additional equations, specifying the properties of the external field, i.e. the algebraic and differential properties of Φ and Ψ. The physical sense of this equation is quite clear: local balance of energy-momentum.

The coordinate free written relationship (12) is equivalent to the following relations: $(\Phi = \alpha^1 \otimes e_1 + \alpha^2 \otimes e_2, \Psi = \alpha^3 \otimes e_3 + \alpha^4 \otimes e_4)$

$$\begin{aligned} &\delta F \wedge *F = \alpha^1 \wedge *F, \ \delta * F \wedge ** F = \alpha^4 \wedge ** F, \\ &\delta F \wedge ** F + \delta * F \wedge *F = \alpha^3 \wedge ** F + \alpha^2 \wedge *F, \end{aligned} \tag{13}$$

or, in components (the 1-forms α^i will be called *currents* also)

$$\begin{aligned} &F_{\mu\nu}(\delta F)^\nu = F_{\mu\nu}(\alpha^1)^\nu, \ (*F)_{\mu\nu}(\delta * F)^\nu = (*F)_{\mu\nu}(\alpha^4)^\nu, \\ &F_{\mu\nu}(\delta * F)^\nu + (*F)_{\mu\nu}(\delta F)^\nu = (*F)_{\mu\nu}(\alpha^3)^\nu + F_{\mu\nu}(\alpha^2)^\nu. \end{aligned} \tag{14}$$

Moving everything on the left, we get

$$(\delta F - \alpha^1) \wedge *F = 0, \ (\delta * F - \alpha^4) \wedge ** F = 0,$$

$$(\delta F - \alpha^3) \wedge ** F + (\delta * F - \alpha^2) \wedge *F = 0,$$

or in components

$$F_{\mu\nu}(\delta F - \alpha^1)^\nu = 0, \ (*F)_{\mu\nu}(\delta * F - \alpha^4)^\nu = 0,$$

$$F_{\mu\nu}(\delta * F - \alpha^2)^\nu + (*F)_{\mu\nu}(\delta F - \alpha^3)^\nu = 0.$$

Summing up the two equations

$$F_{\mu\nu}(\delta F)^\nu = F_{\mu\nu}(\alpha^1)^\nu,\ (*F)_{\mu\nu}(\delta * F)^\nu = (*F)_{\mu\nu}(\alpha^4)^\nu$$

we obtain

$$F_{\mu\nu}(\delta F)^\nu + (*F)_{\mu\nu}(\delta * F)^\nu = \nabla_\nu Q_\mu^\nu = F_{\mu\nu}(\alpha^1)^\nu + (*F)_{\mu\nu}(\alpha^4)^\nu.$$

This relation shows that the sum

$$F_{\mu\nu}(\alpha^1)^\nu + (*F)_{\mu\nu}(\alpha^4)^\nu$$

is a divergence of a 2-tensor, which we denote by $-P_\mu^\nu$. In this way we obtain the local conservation law

$$\nabla_\nu(Q_\mu^\nu + P_\mu^\nu) = 0. \tag{15}$$

Thus, we get the possibility to introduce the full energy-momentum tensor

$$T_\mu^\nu = Q_\mu^\nu + P_\mu^\nu,$$

where P_μ^ν is interpreted as *interaction energy-momentum tensor*. Clearly, P_μ^ν can not be determined uniquely in this way.

So, according to (14), for the 22 functions $F_{\mu\nu}, (\alpha^i)_\mu$ we have 12 equations, and these 12 equations are differential with respect to $F_{\mu\nu}$ and algebraic with respect to $(\alpha^i)_\mu$. Our purpose now is to try to write down differential equations for the components of the 4 currents α^i. The leading idea in pursuing this goal will be to establish a correspondence between the physical concept of *non-dissipation* and the mathematical concept of *integrability of Pfaff system*. The suggestion to look for such a correspondence comes from the following considerations.

Recall from the theory of the ordinary differential equations (or vector fields), that every solution of a system of ordinary differential equations (ODE) defines a local (with respect to the parameter on the trajectory) group of transformations, frequently called *local flow*. This means, in particular, that the motion along the trajectory is admissible in the two directions: we have a *reversible* phenomenon, which has the physical interpretation of *lack of losses* (energy-momentum losses are meant). Assuming this system of ODE describes *fully* the process of motion of a small piece of matter (particle), we

assume at the same time, that *all* energy-momentum exchanges between the particle and the outer field are taken into account, i.e. we have assumed that *there is no dissipation*. In other words, *the physical assumption for the lack of dissipation is mathematically expressed by the existence of a solution - local flow, having definite group properties*. The existence of such a local flow is guaranteed by the corresponding theorem for existence and uniqueness of a solution at given initial conditions. This correspondence between the mathematical fact *integrability* and the physical fact *lack of dissipation* in the simple case "motion of a particle" , we want to generalize in an appropriate way, having in view possible applications in more complicated physical systems, in particular, the physical situation we are going to describe: interaction of the field Ω with some outer field, represented in the exchange process by the four 1-forms α^i. This will allow to write down equations for α^i in a direct way. Of course, in the real world there is always dissipation, and following this idea we are going to take into account its neglecting as conditions (i.e. equations) on the currents α^i. As it is well known, the mathematicians have made serious steps towards study and formulation of criteria for integrability of partial differential equations, so it looks unreasonable to close eyes before the available and represented in appropriate form mathematical results.

Remark. It is interesting, and may be suggesting, to note the following . In physics we have two *universal* things: *dissipation* and *gravitation*. We are going now to establish a correspondence between the physical notion of *dissipation* and the mathematical concept of *non-integrability*. As we know, the mathematical non-integrability is measured by the concept of *curvature*. General theory of Relativity describes gravitation by means of Riemannian curvature. The circle will close if we connect the universal property of any real physical process to dissipate energy-momentum with the only known so far universal interaction in nature, the gravitation.

First we note that our base manifold, where all fields and operations are defined, is the simple 4-dimensional Minkowski space. According to our equations (12) the medium reacts to the field Ω by means of the two $\mathcal{R}^2$-valued 1-forms $\Phi = \alpha^1 \otimes e_1 + \alpha^2 \otimes e_2$ and $\Psi = \alpha^3 \otimes e_1 + \alpha^4 \otimes e_2$. So, we obtain four $\mathcal{R}$-valued 1-forms $\alpha^1, \alpha^2, \alpha^3, \alpha^4$. Because of the 4-dimensions of Minkowski space it is easily seen that only 1-dimensional and 2-dimensional Pfaff systems may be of interest from the Frobenius integrability point of view. All Pfaff systems of higher dimension are trivially integrable.

The integrability equations for 1-dimensional Pfaff systems are

$$\mathbf{d}\alpha^i \wedge \alpha^i = 0,\ i = 1,2,3,4. \tag{16}$$

Every of the 4 equations (16) is equivalent to 4 scalar nonlinear equations for the components of the corresponding 1-form. We note, that the solutions of (16), as well as the solutions of the general integrability equations for a p-dimensional Pfaff system, are determined up to a scalar multiplier, i.e. if α^i define a solution, then $f_i.\alpha^i$ (no summation over i), where f_i are smooth functions, define also a solution.

In case of 2-dimensional Pfaff systems (α^i, α^j), defined by four 1-forms, their maximal number is 4.3=12. The Frobenius equations read

$$\mathbf{d}\alpha^i \wedge \alpha^i \wedge \alpha^j = 0,\ i \neq j. \tag{17}$$

We have here 12 nonlinear equations for the all 16 components of α^i, $i = 1,2,3,4$. Clearly, these equations (17) are substantial only if the corresponding α^i, the exteriour differential $\mathbf{d}\alpha^i$ of which participates in (17), does not satisfy (16) or is not zero.

Our assumption now reads:

Every 2-dimensional Pfaff system, defined by the four 1-forms α^i is completely integrable.

As we mentioned above, physically this assumption means that we neglect the dissipation of energy-momentum. Note also, that 1-dimensional nonintegrable Pfaff systems are admissible, which physically means, that if there is some 1-dimensional nonintegrable Pfaff system among the α^i, e.g. α^1, then the corresponding dissipation of energy-momentum does not flow out of the physical system and it is utilized by the exchange processes, described by the rest currents $\alpha^2, \alpha^3, \alpha^4$.

Finally we note, that (14) and (16) give in general 24 equations for the 22 functions $F_{\mu\nu}, (\alpha)^i_\mu$, which seem to be enough to obtain the dynamics of the system at given initial condition. As for the Maxwell's theory, it corresponds to $\pi_2\delta\Omega = 0$ and $\pi_1\delta\Omega = 4\pi(J_{free} + J_{bound}) \otimes e_1$, i.e. $\alpha^2 = \alpha^4 = 0$ and $\alpha^1 = \alpha^3 = 4\pi(J_{free} + J_{bound})$.

In conclusion we note, that our notion of EM-field requires a simultaneous consideration of a number of soliton-like solutions and, in particular, a possible interaction (interference) among them. This problem does not fit to the goal we set in this paper, so it will be considered somewhere else.

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